

### Spin-0 (scalars) $\phi$

$$\mathcal{L}_{\text{free}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \left(\frac{m c}{\hbar}\right)^2 \phi^2 \quad \text{Notice that the mass term is separate which allows us to consistently handle } m=0 \text{ cases.}$$
$$= \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \left(\frac{m c}{\hbar}\right)^2 \phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0$$

$$\left(\frac{m c}{\hbar}\right)^2 \phi - \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial}{\partial x^\nu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \right) = 0$$

$$\left(\frac{m c}{\hbar}\right)^2 \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\nu \partial_\mu \phi = 0$$

$$\left(\frac{m c}{\hbar}\right)^2 \phi - \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$$

$$\partial_\mu \partial^\mu \phi - \left(\frac{m c}{\hbar}\right)^2 \phi = 0 \quad \text{The Klein-Gordon Equation}$$

Spin-1 (vectors)  $A^\mu$

$\mathcal{L}_{free} = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8\pi} \left(\frac{hc}{k}\right)^2 A^\mu A_\mu$  where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and  $F^{\mu\nu} = \eta^{\mu\lambda} \eta^{\nu\rho} F_{\lambda\rho}$

$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu} \right) = 0$

$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{4\pi} \left(\frac{hc}{k}\right)^2 A^\mu$

$\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu} = \frac{1}{4\pi} F^{\mu\nu}$  (You get to fill in the details in your homework)

Then:  $\partial_\mu F^{\mu\nu} - \left(\frac{hc}{k}\right)^2 A^\nu = 0$  The Proca Equation

Taking  $m^2 = 0$  we have  $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \left. \begin{aligned} \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 0 \end{aligned} \right\} \text{2 of Maxwell's equations}$

Note:  $F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$

$\left. \begin{aligned} \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned} \right\} \text{Non-dynamical, come from electromagnetic "geometry"}$

Spin-1/2 (spinors)  $\psi$  &  $\bar{\psi}$

$\mathcal{L}_{\text{free}} = (kc)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$  We treat  $\psi$  &  $\bar{\psi}$  as independent degrees of freedom for reasons to be discussed.

Varying w.r.t.  $\bar{\psi}$ :

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \right) = 0$$

$$\underbrace{kc\gamma^\mu\psi + mc\psi}_{\equiv \not{D}} = 0 \Rightarrow \not{D}\psi + \frac{mc}{\hbar}\psi = 0 \quad \text{The Dirac Equation}$$

Varying w.r.t.  $\psi$ :

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) = 0$$

$$+mc\bar{\psi} - \partial_\mu (kc\bar{\psi}\gamma^\mu) = 0 \Rightarrow \not{D}\bar{\psi} - \frac{mc}{\hbar}\bar{\psi} = 0 \quad (\text{The adjoint of the Dirac Equation})$$

↑  
Interesting...

Consider the 3 equations of motion we have discussed so far:

Spin 0:  $\partial_\mu \partial^\mu \phi - \left(\frac{m_0 c}{\hbar}\right)^2 \phi = 0$  K-G equation

Spin 1/2:  $\not{\partial} \psi + \frac{m_0 c}{\hbar} \psi = 0 \Rightarrow$  You showed in your HW that  $\Rightarrow \partial_\mu \partial^\mu \psi - \left(\frac{m_0 c}{\hbar}\right)^2 \psi = 0$  K-G equation

Spin 1:  $\partial_\mu F^{\mu\nu} - \left(\frac{m_0 c}{\hbar}\right)^2 A^\nu = 0 \Rightarrow \underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=0 \text{ since sym-antisym}} - \left(\frac{m_0 c}{\hbar}\right)^2 \underbrace{\partial_\nu A^\nu}_{=0} = 0$  holds for each component of  $\psi$  and  $A^\nu$  separately!  
 $\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) - \left(\frac{m_0 c}{\hbar}\right)^2 A^\nu = \underbrace{\partial_\mu \partial^\mu A^\nu - \left(\frac{m_0 c}{\hbar}\right)^2 A^\nu}_{\text{K-G equation}} = 0$

Why does everything also satisfy the Klein-Gordon equation?

Consider:  $\frac{E^2}{c^2} - p^2 = m_0^2 c^2 \Rightarrow p_\mu p^\mu + m_0^2 c^2 = 0$  let  $p_\mu = i\hbar \partial_\mu$  and act on  $\phi$

$- \hbar^2 \partial_\mu \partial^\mu \phi + m_0^2 c^2 \phi = 0 \Rightarrow \underbrace{\partial_\mu \partial^\mu \phi - \left(\frac{m_0 c}{\hbar}\right)^2 \phi}_{\text{K-G equation}} = 0$

Note: Starting w/  $\frac{p^2}{2m} + V = E$   
 using  $p \rightarrow -i\hbar \vec{\nabla}$ ,  $E \rightarrow i\hbar \frac{\partial}{\partial t}$   
 then  $-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}$   
 Non-relativistic Time-dependent Schrodinger Equation

K-G equation is really just reflecting the mass-shell condition which all real degrees of freedom must satisfy!

So why do we have Dirac and Proca? What more do they tell us?

To understand this we need to think about degrees of freedom. Typically the # d.o.f. is equal to the dimension of configuration space or half the dimension of phase space.

Example: 1 non-relativistic spin 0 particle is described by:  $x(t), y(t), z(t) \Rightarrow$  3D configuration space  
or:  $x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)$  6D phase space  
 $\Rightarrow$  3 d.o.f. ( $3N$  for  $N$  particles)

For fields the counting is more subtle. Technically a field has an  $\infty$  # of d.o.f. since we must specify the field value at each spacetime point. However we know that the "motion" of particle-like field fluctuations is covered by K-G equation, so we can set that aside and ask how many d.o.f. per point are left over. For spin 0 scalars the K-G equation does it all.

But what if a particle has nontrivial internal spin? Long ago in a land far away, Eugene Wigner developed a method for identifying the degrees of freedom using some of the deeper elements of group theory (see "Wigner's classification" or "induced representations"). I will try to give you a sense of the result.

The main idea (in physics) is that the "motion" of a particle, captured by the 4-momentum  $P^\mu$ , partly characterizes the freedom in specifying what a particle is doing (its d.o.f.s). The question then is "after specifying  $P^\mu$ , what else can be chosen without changing  $P^\mu$ ."

The good news is that counting d.o.f. is independent of reference frame (as it should be in relativity) so we can use the simplest one.

Consider a particle in its rest frame:  $P_{\text{rest}}^\mu = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$  } The transformations which leave this invariant are clearly 3D rotations leaving  $\vec{p} = \vec{0}$  unchanged.

It is not intuitive, but we have just as much freedom left over in any other frame.

The upshot is that to characterize the internal spin states, we get to carry over all of the familiar angular momentum information from 3D NR QM.

So let's review some essential results for angular momentum in 3D NR QM.

First of all, in contrast to linear momentum, angular momentum is always discrete in QM. recall this is continuous for  $\infty$  dimensions and discrete for finite dimensions

The reason is that the angle through which we rotate is compact, i.e.  $\theta \in (0, 2\pi]$ . (like particle in a box)

For quantized internal spin, the total spin magnitude  $S^2$  is fixed once and for all. But we can dynamically change  $S_z$ . This is not true for orbital angular momentum where  $L^2$  and  $L_z$  can change.

From QM you know that if:  $S \sim 0 \Rightarrow S_z = 0$  1 d.o.f.  
 $S \sim \frac{1}{2} \Rightarrow S_z = \pm \frac{1}{2} \hbar$  2 d.o.f.  
 $S \sim 1 \Rightarrow S_z = (-1, 0, +1)\hbar$  3 d.o.f.  
etc.

(actually,  $S^2 = s(s+1)\hbar^2$ )  
Note:  $S^2 > S_z^2 = s^2 \hbar^2$

We can never have  $S = S_z$  since then  $S_x = S_y = 0$ , but we know these 3 cannot be simultaneously known.

As an example consider  $A^\nu$  and the Proca equation. Naively  $A^\nu$  should have 4 real parameters to be specified, but being spin 1, we should really only expect 3 (+1, 0, -1).

$$\partial_\mu F^{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0 \Rightarrow \begin{cases} \partial_\mu \partial^\mu A^\nu + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0 & \text{K-G} \\ \partial_\nu A^\nu = 0 & \text{One real constraint removing 1 d.o.f.} \end{cases}$$

As for spinors  $\psi$  and the Dirac equation, we know that naively  $\psi$  has 4 complex components or 8 real degrees of freedom. But a spin  $\frac{1}{2}$  particle should only have 2 ( $\pm \frac{1}{2}$ ) so the Dirac equation must do a lot of cutting down.

To see this consider the Dirac equation:  $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar} \psi = 0$  in momentum representation ( $i\hbar \partial_\mu \rightarrow p_\mu$ )

$$-i \gamma^\mu p_\mu \psi + mc \psi = 0$$

In the rest frame  $p_\mu = (-mc, 0, 0, 0)$  so this becomes:

$$i \gamma^0 mc \psi + mc \psi = 0$$

$$(i \gamma^0 + 1) mc \psi = 0$$

But:  $(i \gamma^0 + 1)^2 = (-\gamma^0 + 2i \gamma^0 + 1) = 2(i \gamma^0 + 1) \Rightarrow i \gamma^0 + 1$  is a projection operator from  $8 \rightarrow 4$  d.o.f.

But wait, didn't we expect to get down to 2?